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THE SUBINDEPENDENCE OF COORDINATE SLABS IN l_p^n BALLS

BY

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ABSTRACT

It is proved that if the probability P is normalised Lebesgue measure on one of the l_p^n balls in $\mathbf{R}^n$, then for any sequence $t_1, t_2, \ldots, t_n$ of positive numbers, the coordinate slabs $\{|x_i| \le t_i\}$ are subindependent, namely,

$$P\left(\bigcap_1^n \{|x_i| \le t_i\}\right) \le \prod_1^n P(\{|x_i| \le t_i\}).$$

A consequence of this result is that the proportion of the volume of the l_1^n ball which is inside the cube $[-t,t]^n$ is less than or equal to $f_n(t) = (1-(1-t)^n)^n$. It turns out that this estimate is remarkably accurate over most of the range of values of t. A reverse inequality, demonstrating this, is the second major result of the article.

1. Introduction

Schechtman and Zinn, in [1], proved that the proportion of the volume left in the l_p^n ball after removing a t-multiple of the l_q^n ball is of order $\exp(-cnt^p)$ when $p < q$. Recall that the unit l_p^n ball which is denoted B_p^n is the set $\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n\colon \sum_{i=1}^n |x_i|^p \le 1\}$. Taking limits as $q \longrightarrow \infty$, they also mention some results about the proportion of the volume of the l_p^n ball

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which is outside the cube $[-t,t]^n$. If P is as in the abstract, their results in this particular case are:

if $t \geq \tau\left((\log n)/n\right)^{1/p}$, then $P(\{\|x\|_\infty \geq t\}) \leq \exp\left(-\gamma n t^p/p\right)$, and

if $2/n^{1/p} \leq t \leq \frac{1}{2}$, then $P(\{\|x\|_\infty \geq t\}) \geq \exp\left(-\Gamma n t^p/p\right)$,

where γ, Γ and τ are universal constants.

We consider only the case $q = \infty$ here, but our results are much stronger. For simplicity we shall illustrate this only in the case $p = 1$ although the most important result will be described for all p. This result is the subindependence of coordinate slabs, stated below as Theorem 1.

THEOREM 1 (Subindependence of coordinate slabs): *If the probability P is normalised Lebesgue measure on one of the l_p^n balls in $\mathbf{R}^n$, then for any sequence $t_1, \ldots, t_n$ of positive numbers,*

$$P\left(\bigcap_1^n \{|x_i| \leq t_i\}\right) \leq \prod_1^n P(\{|x_i| \leq t_i\}).$$

The particular case $p = 1$, $t_1 = \cdots = t_n$ of Theorem 1 gives an upper bound for the proportion of the volume of the l_1^n ball which is inside the cube $[-t,t]^n$. Since the proportion of the volume of the l_1^n ball which is inside a coordinate slab of width $2t$ is $1-(1-t)^n$ when $t \leq 1$, the result in this case is given by the following Corollary.

COROLLARY 1: *If $F_n(t)$ is the proportion of the volume of the l_1^n ball inside the cube $[-t,t]^n$, then*

$$F_n(t) \leq f_n(t) = (1-(1-t)^n)^n.$$

Although $F_n(t)$ is the function $\sum_0^{[1/t]}(-1)^j\binom{n}{j}(1-jt)^n$, which is a spline with many knots, we prove in Theorem 2 that the polynomial $f_n(t) = (1-(1-t)^n)^n$ is an astonishingly good approximation to $F_n(t)$, at least when $F_n(t)$ is not too small.

THEOREM 2 (An estimate in the reverse direction): *With $F_n(t)$ as above,*

$$\frac{1-F_n(t)}{1-f_n(t)} = 1 + O\left(\frac{(\log n)^3}{n}\right)$$

as $n \to \infty$, uniformly in t.

Theorem 2 enables us to describe the threshold behaviour of $F_n(t)$ much more precisely than Schechtman and Zinn. For example, if $t = (\log n - \log c)/n$ then the information we get from Theorem 2 is that $F_n(t)$ should be something like

$f_n(t)$, which in turn is something like $(1-\exp(-\log n+\log c))^n = (1-c/n)^n \simeq \exp(-c)$.

2. Method

In this section we will briefly explain the crucial points of the proofs of these two Theorems for the simplest case when $p=1$ and $t_1=\cdots=t_n=t$.

The proof of Theorem 1 (the upper bound for F_n) depends on a very convenient interaction between two different equations expressing F_n and its derivative in terms of F_{n-1}. Each of these equations is proved using a simple geometric argument: they can readily be combined to give a differential inequality for F_n which integrates up to the stated result.

These equations are

$$F_n(t) = n\int_0^t (1-u)^{n-1}F_{n-1}\left(\frac{t}{1-u}\right)du,$$
$$\frac{d}{dt}F_n(t) = n^2(1-t)^{n-1}F_{n-1}\left(\frac{t}{1-t}\right).$$

The upper bound is extremely precise as long as $F_n(t)$ is not too small. The easiest way to state this is to write it as an estimate for the volume outside the cube, namely for $1-F_n(t)$. This is what we do in Theorem 2.

The proof of Theorem 2 (a lower bound for F_n) is technically more complicated although it is much less delicate. The crucial point is to show that at its maximum, the function $(1-F_n)/(1-f_n)$ is dominated by the value of a related function, which in turn can be shown to be small by means of the (rather precise) upper bound already proved.

In fact, this related function, say $G_n(t)$, is not as small as we would like it to be in the whole interval $(0,1)$, but it behaves nicely in a smaller interval $[t_n, 1/2]$, for some value of t_n which is roughly like $(\log n - \log\log n)/n$. It is in this range that $(1-F_n)/(1-f_n)$ actually attains its maximum. However, for technical reasons, it is simpler to show directly that $(1-F_n)/(1-f_n)$ is small outside this interval.

3. The upper bound

In this section we shall give a detailed proof for the simplest case of Theorem 1, $p=1$. The other cases are simple generalizations of this one, so only a brief sketch of the proof will be given then.

THEOREM 1 (Subindependence of coordinate slabs): *If the probability P is normalised Lebesgue measure on one of the l_p^n balls in $\mathbf{R}^n$, then for any sequence $t_1, \ldots, t_n$ of positive numbers,*

$$P\left(\bigcap_1^n \{|x_i| \le t_i\}\right) \le \prod_1^n P(\{|x_i| \le t_i\}).$$

Proof of Theorem 1 for the case $p = 1$, $t_1 = \cdots = t_n = t$: Except in the trivial case $t \ge 1$ the problem is to show that the proportion of the volume of the unit l_1^n ball which is inside the cube $Q_n(t) = [-t,t]^n$ is bounded from above by the function $f_n(t) = (1-(1-t)^n)^n$. This proportion will be denoted by $F_n(t)$.

The proof uses the following two equations:

$$F_n(t) = n\int_0^t (1-u)^{n-1} F_{n-1}\left(\frac{t}{1-u}\right) du, \tag{3.1}$$

$$\frac{d}{dt}F_n(t) = n^2(1-t)^{n-1}F_{n-1}\left(\frac{t}{1-t}\right). \tag{3.2}$$

Since F_{n-1} is an increasing function, $F_{n-1}\left(\frac{t}{1-u}\right)$ is increasing in u. So from (3.1) we get

$$F_n(t) \le nF_{n-1}\left(\frac{t}{1-t}\right)\int_0^t (1-u)^{n-1}du.$$

For convenience, we shall abbreviate the integral

$$\int_0^t (1-u)^{n-1}du = \frac{1-(1-t)^n}{n}$$

by $Y_n(t)$. Then (3.2) and the inequality can be written

$$F_n(t) \le nF_{n-1}\left(\frac{t}{1-t}\right)Y_n(t), \tag{3.3}$$

$$\frac{d}{dt}F_n(t) = n^2F_{n-1}\left(\frac{t}{1-t}\right)\frac{d}{dt}Y_n(t). \tag{3.4}$$

If we eliminate the factor $nF_{n-1}\left(\frac{t}{1-t}\right)$ we get

$$\frac{\frac{d}{dt}F_n(t)}{F_n(t)} \ge n\frac{\frac{d}{dt}Y_n(t)}{Y_n(t)} \tag{3.5}$$

and, by integrating from t to 1, we get the desired result

$$F_n(t) \le \left(\frac{Y_n(t)}{Y_n(1)}\right)^n = (1-(1-t)^n)^n.$$

It remains to prove the relations (3.1) and (3.2).

For the first one, let $H_u = \{x \in \mathbf{R}^n : x_1 = u\}$. Then

$$\begin{aligned}
F_n(t) &= \frac{\mathrm{Vol}_n(Q_n(t) \cap B_1^n)}{\mathrm{Vol}_n(B_1^n)} \\
&= \frac{n!}{2^n} 2 \int_0^t \mathrm{Vol}_{n-1}(Q_n(t) \cap B_1^n \cap H_u) du \\
&= \frac{n!}{2^{n-1}} \int_0^t \mathrm{Vol}_{n-1}(Q_{n-1}(t) \cap B_1^{n-1}(1-u)) du \\
&= n \int_0^t (1-u)^{n-1} \frac{\mathrm{Vol}_{n-1}(Q_{n-1}(t) \cap B_1^{n-1}(1-u))}{\mathrm{Vol}_{n-1}(B_1^{n-1}(1-u))} du \\
&= n \int_0^t (1-u)^{n-1} F_{n-1}\left(\frac{t}{1-u}\right) du.
\end{aligned}$$

For the second one, put $H_n(t) = \mathrm{Vol}_n(Q_n(t) \cap B_1^n)$. Since

$$F_n(t) = \frac{H_n(t)}{\mathrm{Vol}_n(B_n^1)} = \frac{n!}{2^n} H_n(t),$$

to find $\frac{d}{dt} F_n(t)$ it suffices to find $\frac{d}{dt} H_n(t)$, which is

$$\begin{aligned}
\frac{d}{dt} H_n(t) &= \lim_{h \to 0} \frac{H_n(t+h) - H_n(t)}{h} \\
&= 2n \, \mathrm{Vol}_{n-1}(Q_{n-1}(t) \cap B_1^{n-1}(1-t))
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{d}{dt} F_n(t) &= n^2 \frac{\mathrm{Vol}_{n-1}(Q_{n-1}(t) \cap B_1^{n-1}(1-t))}{\mathrm{Vol}_{n-1}(B_1^{n-1})} \\
&= n^2 (1-t)^{n-1} \frac{\mathrm{Vol}_{n-1}(Q_{n-1}(t) \cap B_1^{n-1}(1-t))}{\mathrm{Vol}_{n-1}(B_1^{n-1}(1-t))} \\
&= n^2 (1-t)^{n-1} F_{n-1}\left(\frac{t}{1-t}\right).
\end{aligned}$$

Proof of Theorem 1 for the case $p = 1$: For convenience, let $F_n(t_1, \dots, t_n)$ denote the proportion of the volume of the unit l_1^n ball which is inside the cuboid $Q_n(t_1, \dots, t_n) = [-t_1, t_1] \times \cdots \times [-t_n, t_n]$. The Theorem states that

$$F_n(t_1, \dots, t_n) \le \frac{Y_n(t_1)}{Y_n(1)} \cdots \frac{Y_n(t_n)}{Y_n(1)}$$

where $Y_n(t)$ is the integral $\int_0^{\min\{1,t\}} (1-u)^{n-1} du$.

Of course, if one of the t_i's is zero, then $F_k(t_1,\dots,t_k)=0$ and the inequality is trivial. It is also trivial when all the t_i's are greater than 1.

If neither of these trivial cases applies, we prove that as long as for some i, t_i is less than 1, the value of the function F_n at point $(t_1,\dots,t_n)$ is dominated by an appropriate multiple of the value of F_n, at the point with the ith coordinate replaced by 1 and the rest remaining the same, i.e.

$$F_n(t_1,\dots,t_n)\le \frac{Y_n(t_i)}{Y_n(1)}F_n(t_1\dots,t_{i-1},1,t_{i+1}\dots,t_n). \tag{3.6}$$

So, if we suppose, without loss of generality, that $0<t_i<1$ for $i=1,\dots,k$ $(1<k\le n)$ and $t_i\ge 1$ for $i=k+1,\dots,n$, then we will have in turn the following inequalities:

$$\begin{aligned}F_n(t_1,\dots,t_n)&\le \frac{Y_n(t_1)}{Y_n(1)}F_n(1,t_2,\dots,t_n)\\&\le \frac{Y_n(t_1)}{Y_n(1)}\frac{Y_n(t_2)}{Y_n(1)}F_n(1,1,t_3,\dots,t_n)\\&\dots\dots\dots\dots\dots\dots\\&\le \frac{Y_n(t_1)}{Y_n(1)}\cdots\frac{Y_n(t_k)}{Y_n(1)}F_n(1,\dots,1,t_{k+1},\dots,t_n).\end{aligned}$$

Since $F_n(1,\dots,1,t_{k+1}\dots,t_n)=1$ the proof is complete.

Thus, the crucial point is to prove (3.6). Without loss of generality, we will prove this for $i=1$, namely the relation

$$F_n(t_1,\dots,t_n)\le \frac{Y_n(t_1)}{Y_n(1)}F_n(1,t_2,\dots,t_n) \tag{3.7}$$

when $0<t_1<1$.

To do this, we again combine two equations. The first one relates F_n and F_{n-1}, and the second one relates F_{n-1} and the partial derivative of F_n with respect to the first coordinate, at point t_1. These are

$$\begin{aligned}F_n(t_1,\dots,t_n)&=n\int_0^t(1-u)^{n-1}F_{n-1}\left(\frac{t_2}{1-u},\dots,\frac{t_n}{1-u}\right)du\\&\le nF_{n-1}\left(\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)\int_0^t(1-u)^{n-1}du\\&=nF_{n-1}\left(\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)Y_n(t_1)\end{aligned} \tag{3.8}$$

and

$$\frac{\partial}{\partial t_1}F_n(t_1,\dots,t_n)=n(1-t_1)^{n-1}F_{n-1}\left(\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right) \tag{3.9}$$

$$= F_{n-1}\left(\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)\frac{d}{dt_1}Y_n(t_1).$$

Eliminating $F_{n-1}\left(\frac{t_2}{1-t_1},\dots,\frac{t_n}{1-t_1}\right)$, we get

$$\frac{\frac{\partial}{\partial t_1}F_n(t_1,\dots,t_n)}{F_n(t_1,\dots,t_n)} \geq \frac{\frac{d}{dt_1}Y_n(t_1)}{Y_n(t_1)} \tag{3.10}$$

which integrates to (3.7).

The proofs of (3.8) and (3.9) are very similar to the proofs of (3.1) and (3.2).

Sketch of the proof of Theorem 1 for the case $p>1$, $t_1=\dots=t_n=t$: Since the proof of this case does not differ too much from the one given for the first case, we shall only write the two basic equations that are used in place of (3.1) and (3.2). A slightly different notation is used here. $Y_n^p(t)$ stands for $\int_0^{\min\{1,t\}}(1-u^p)^{(n-1)/p}du$, v_n^p for the volume of the B_p^n ball, and $F_n^p(t)$ for the proportion of the B_p^n ball, which is inside the cube $Q_n(t)$.

The relations are as follows:

$$\begin{aligned} F_n^p(t) &= \frac{2v_{n-1}^p}{v_n^p}\int_0^t (1-u^p)^{(n-1)/p}F_{n-1}^p\left(\left(\frac{t^p}{1-u^p}\right)^{1/p}\right)du \\ &\leq \frac{2v_{n-1}^p}{v_n^p}F_{n-1}^p\left(\left(\frac{t^p}{1-t^p}\right)^{1/p}\right)Y_n^p(t) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \frac{d}{dt}F_n^p(t) &= \frac{2nv_{n-1}^p}{v_n^p}(1-t^p)^{(n-1)/p}F_{n-1}^p\left(\left(\frac{t^p}{1-t^p}\right)^{1/p}\right) \\ &= \frac{2nv_{n-1}^p}{v_n^p}F_{n-1}^p\left(\left(\frac{t^p}{1-t^p}\right)^{1/p}\right)\frac{d}{dt}Y_n^p(t). \end{aligned} \tag{3.12}$$

Remarks: (1) (3.5) and (3.10) actually state that the functions $F_n(t)/(Y_n(t))^n$ and $F_n(t,t_2,\dots,t_n)/Y_n(t)$ are increasing in t.

A consequence of this is that the function $F_n(t_1,\dots,t_n)/Y_n(t_1)\cdots Y_n(t_n)$ is increasing in each coordinate.

(2) If $0<t_i<1$ for $i=1,\dots,k$ $(1<k\leq n)$ and $t_i\geq 1$ for $i=k+1,\dots,n$, then Theorem 1 states that $F_n(t_1,\dots,t_n)\leq(1-(1-t_1)^n)\cdots(1-(1-t_k)^n)$.

4. The lower bound

Using the notation introduced in the previous section, we shall prove that the function $f_n(t)$ is not only an upper bound (see Theorem 1), but it is also a very good approximation to $F_n(t)$, within the interesting range of t. More precisely, we prove that the function $(1-F_n(t))/(1-f_n(t))$ converges to 1 uniformly in t, as stated in the next Theorem.

THEOREM 2 (An estimate in the reverse direction):

$$\frac{1-F_n(t)}{1-f_n(t)} = 1 + O\left(\frac{(\log n)^3}{n}\right)$$

uniformly in t.

We focus our attention on the point $t_{\max}$, where $(1-F_n(t))/(1-f_n(t))$ attains its maximum value. In the first Lemma below, we find a function $G_n(t)$ which dominates $(1-F_n(t))/(1-f_n(t))$ at $t_{\max}$. This related function is proved to be small in a particular range, where $t_{\max}$ actually occurs. Outside this range $(1-F_n(t))/(1-f_n(t))$ is small for very simple reasons. To avoid technical difficulties, we don't actually prove that $t_{\max}$ is in this particular range.

LEMMA 1: *At its maximum point, the function $(1-F_n(t))/(1-f_n(t))$ is dominated by the value of the function*

$$G_n(t) = \begin{cases} \left[\frac{1-\left(1-\frac{t}{1-t}\right)^{n-1}}{1-(1-t)^n}\right]^{n-1}, & 0 < t \le 1/2, \\ [1-(1-t)^n]^{-(n-1)}, & 1/2 < t < 1. \end{cases}$$

Proof of Lemma 1: Before embarking upon the proof it is perhaps worth mentioning that it depends critically upon Theorem 1 (the upper bound for F_n) already proved.

It is easy to check that $(1-F_n(t))/(1-f_n(t)) \to 1$ as $t \to 0$ or $t \to 1$. So $(1-F_n)/(1-f_n)$ attains its maximum in (0,1). So

$$\left[\frac{d}{dt}\log\left(\frac{1-F_n(t)}{1-f_n(t)}\right)\right]_{t_{\max}} = 0,$$

i.e.

$$\frac{1-F_n(t_{\max})}{1-f_n(t_{\max})} = \frac{\frac{d}{dt}F_n(t_{\max})}{\frac{d}{dt}f_n(t_{\max})}.$$

But $\frac{d}{dt}F_n(t)$ has already been calculated in (3.2). Substituting this in the above relation, as well as $\frac{d}{dt}f_n(t_{\max})$, we get that

$$\frac{1-F_n(t_{\max})}{1-f_n(t_{\max})} = \frac{F_{n-1}\left(\frac{t_{\max}}{1-t_{\max}}\right)}{(1-(1-t_{\max})^n)^{n-1}}.$$

Of course,

$$F_{n-1}\left(\frac{t_{\max}}{1-t_{\max}}\right)=1 \quad \text{if } 1/2<t_{\max}<1.$$

To prove the required inequality for $0<t_{\max}\le 1/2$, it is sufficient to apply Theorem 1 in order to dominate $F_{n-1}\left(\frac{t_{\max}}{1-t_{\max}}\right)$. Thus we get

$$\frac{1-F_n(t_{\max})}{1-f_n(t_{\max})}\le\left[\frac{1-\left(1-\frac{t_{\max}}{1-t_{\max}}\right)^{n-1}}{1-(1-t_{\max})^n}\right]^{n-1}=G_n(t_{\max}).$$

Proof of Theorem 2: As we have already mentioned, for technical reasons, we shall divide the interval (0,1) into three parts, and we will examine separately the possibilities that $t_{\max}$ occurs in each of these parts.

More precisely, choose t_n such that $(1-t_n)^n=(\log n)/n$ and consider the intervals $(0,t_n)$, $\left[t_n,\frac{1}{2}\right]$ and $\left(\frac{1}{2},1\right)$.

t_n is something like $(\log n-\log\log n)/n$ and is certainly less than $(\log n)/n$.

Numerical evidence indicates that $t_{\max}$ is about $(\log n)/n$ but we eliminate the other intervals directly.

We shall prove that for $t\in\left(\frac{1}{2},1\right)$,

$$\frac{1-F_n(t)}{1-f_n(t)}\le 1+\frac{1}{n}.$$

It is quite easy to calculate that $F_n(t)=1-n(1-t)^n$ when $t\in\left(\frac{1}{2},1\right)$ by integrating (3.2) where $F_{n-1}\left(\frac{t}{1-t}\right)=1$.

So, the inequality we want to prove becomes

$$\frac{n(1-t)^n}{1-(1-(1-t)^n)^n}\le 1+\frac{1}{n}.$$

If we put $s=(1-t)^n$ (so that $s\le 1/2^n$), the problem is to check that

$$\frac{ns}{1-(1-s)^n}\le 1+\frac{1}{n},$$

i.e. that

$$(1-s)^n\le 1-\frac{n^2}{n+1}s,$$

which is certainly true if $s\le 1/2^n$.

We shall prove that for all t in $(0,t_n)$ not only is the function $(1-F_n(t))/(1-f_n(t))$ close to 1, but so is the function $(1-f_n(t))^{-1}$.

Since f_n is increasing,

$$\begin{aligned} f_n(t) &= (1-(1-t)^n)^n \\ &\le (1-(1-t_n)^n)^n \\ &= \left(1 - \frac{\log n}{n}\right)^n \\ &\le \exp(-\log n) = \frac{1}{n}. \end{aligned}$$

Hence

$$\frac{1}{1-f_n} = 1 + O\left(\frac{1}{n}\right).$$

Finally we study $F_n(t)$ for $t \in [t_n, \frac{1}{2}]$. By Lemma 1,

$$\frac{1-F_n(t_{\max})}{1-f_n(t_{\max})} \le G_n(t_{\max}).$$

We shall prove that $G_n(t)$ is as small as required in the range $t \in [t_n, \frac{1}{2}]$, namely that

$$G_n(t) \le 1 + O\left(\frac{(\log n)^3}{n}\right).$$

By the first estimate in Lemma 1, G_n in this range is

$$G_n(t) = \left[1 + \frac{(1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n-1}}{1-(1-t)^n}\right]^{n-1}.$$

Thus, it is enough to prove that

$$\frac{(1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n-1}}{1-(1-t)^n} \le O\left(\frac{(\log n)^3}{n^2}\right).$$

Indeed, since the factor $1-(1-t)^n$ is like a constant in this interval, it suffices to show that $(1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n-1}$ is dominated by the decreasing function $n(1-t)^{n-2}t^2$ (decreasing for $t \ge 2/n$), which at t_n is as small as required.

However,

$$\begin{aligned}(1-t)^n - \left(1-\frac{t}{1-t}\right)^{n-1} &\le (1-t)^n - \left(1-\frac{t}{1-t}\right)^n \\ &= \int_{1-\frac{t}{1-t}}^{1-t} nu^{n-1}du \\ &\le \frac{t^2}{1-t}n(1-t)^{n-1} \\ &= n(1-t)^{n-2}t^2 \\ &\le n(1-t_n)^{n-2}t_n^2 \\ &\le 2n(1-t_n)^n t_n^2 \\ &\le 2n\frac{\log n}{n}\frac{(\log n)^2}{n^2} \\ &= O\left(\frac{(\log n)^3}{n^2}\right),\end{aligned}$$

which completes the proof.

Note added in proofs: The authors recently found a shorter proof of Theorem 1 which replaces many of the formulae with a rearrangement argument.

Reference

[1] G. Schechtman and J. Zinn, *On the volume of the intersection of two L_p^n balls*, Proceedings of the American Mathematical Society **110** (1990), 217–224.